

The Turán number of a tree

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A famous conjecture of Erdős and Sós is that every graph with average degree larger than $k - 2$ contains every tree on k vertices as a subgraph. A very nice proof of this conjecture is found by GPT-6 Astra. I present Riordan and Scott interpretation of the proof (arXiv 2609.15893). Let $|T|$ denote the number of vertices of a tree T and $d = d(G)$ denote the average degree of a graph G with n vertices. An edge (v_1, v_i) of G in a vertex ordering $v_1, \dots, v_n$ is called T -jumping if G contains a copy of T rooted at v_1 having all vertices in $\{v_1, \dots, v_{i-1}\}$. Let $X = X(G, T)$ be the random variable for the number of T -jumping edges in the probability space of vertex orderings π of G taking each with probability $\frac{1}{n!}$. The proof of the conjecture is based on the following lower bound on $\mathbb{E}(X)$, the expectation of X .

Theorem 1. (*GPT-6 Astra*) $\mathbb{E}(X) \geq d - |T| + 1$.

Theorem 1 implies the E-S conjecture, applying it to the tree A obtained by removing a degree one vertex $p \in V(T)$ and defining the root of A as the neighbor of p in T . If G has no copy of T , then there are no A -jumping edges in any ordering of $V(G)$. Thus, by Theorem 1,

$$\mathbb{E}(X) = 0 \geq d - |A| + 1 = d - |T| + 2,$$

implying $d \leq |T| - 2$, contradicting the assumption of the conjecture.